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CONVEXITY, SMOOTHNESS AND MARTINGALE INEQUALITIES

BY

D. J. H. GARLING

ABSTRACT

Necessary and sufficient conditions are given, in terms of the behaviour of martingales, for a Banach space to be given on equivalent norm under which it is δ-uniformly convex or ρ-uniformly smooth, where δ and ρ are suitable Orlicz functions.

1. Introduction

Among many other interesting results, Pisier [3], [4] has shown that, if $2 \leqq q < \infty$, a Banach space X can be given an equivalently q-uniformly convex norm if and only if there exists a constant C such that

$$E(\|x_0\|^q) + \sum_{n=1}^{\infty} E(\|x_n - x_{n-1}\|^q) \leqq C^q \sup_n E(\|x_n\|^q)$$

for all X-valued martingales (and that it is sufficient for the condition to hold for Walsh–Paley martingales); a similar characterization is given of Banach spaces which can be given an equivalent p-uniformly smooth norm (where $1 < p \leqq 2$). In this paper we shall extend these results, first by considering δ-uniform convexity and ρ-uniform smoothness, where δ and ρ are suitable Orlicz functions, and secondly by obtaining conditions in terms of uniformly bounded martingales (for uniform convexity) and L^1-convergence (for uniform smoothness).

We gather together some rather elementary remarks about martingales in section 3.

The main theorems are established in sections 4 and 5. The results concerning uniform convexity are obtained directly; we prove results concerning uniform smoothness by duality.

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2. Terminology and notation

We use the customary terminology (as employed by Lindberg [2], for example) for Orlicz functions and Orlicz spaces. If f and g are continuous non-decreasing functions on $[0,2]$, with $f(0)=g(0)=0$, we say that $f \overset{o}{<} g$ if there exists $0<\lambda \leqq 1$ such that $\lambda f(\lambda x) \leqq g(x)$ for all x in $[0,2]$, and say that $f \overset{o}{\sim} g$ if $f \overset{o}{<} g$ and $g \overset{o}{<} f$.

We recall that if $(X, \| \ \|)$ is a Banach space, the *modulus of convexity* δ_X is defined by

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2}: \|x\|=\|y\|=1, \|x-y\|=\varepsilon\right\}$$

for $0 \leqq \varepsilon \leqq 2$. X is *uniformly convex* if $\delta_X(\varepsilon)>0$ for $\varepsilon>0$. If δ is an Orlicz M-function on $[0,2]$ we say that X is *δ-uniformly convex* if there exists $k>0$ such that

$$\|\tfrac{1}{2}(x+y)\|+k\delta(k\|x-y\|) \leqq 1$$

whenever $\|x\| \leqq 1$ and $\|y\| \leqq 1$. Thus X is δ-uniformly convex if and only if $\delta_X \overset{o}{>} \delta$. If X is δ-uniformly convex, where $\delta(\varepsilon)=\varepsilon^p$, we say that X is *p-uniformly convex.*

We recall also that the *modulus of smoothness* ρ_X is defined by

$$\rho_X(\tau)=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1: \|x\|=1, \|y\| \leqq \tau\right\}.$$

X is *uniformly smooth* if $\rho_X(\tau)=o(\tau)$ at 0. If ρ is an Orlicz M-function on $[0,\infty)$, we say that X is *ρ-uniformly smooth* if there exists $K>0$ such that

$$\|x+y\|+\|x-y\| \leqq 2+K\rho(K\|y\|)$$

whenever $\|x\|=1$. Thus X is ρ-uniformly smooth if and only if $\rho_X \overset{o}{<} \rho$. If X is ρ-uniformly smooth, where $\rho(\tau)=\tau^p$, we say that X is *p-uniformly smooth.*

We shall require the following two fundamental properties of the modulus of convexity (cf. [1, corol. 11, prop. 19 and the remarks on p.138]):

(i) For each Banach space X there exists a function δ on $[0,2]$ such that $\delta \overset{o}{\sim} \delta_X$ and $\delta(\varepsilon^{1/2})$ is convex;

(ii) If $1 < p \leqq 2$, $\delta_{L^p(X)} \overset{\circ}{\sim} \delta_X$;
further the constants of equivalence do not depend upon X.

As far as martingales are concerned, we shall suppose that $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is an increasing sequence of sub-σ-fields of a probability space (Ω, F, P), with $F_0 = (\Omega, \emptyset)$, and with F the σ-field generated by the sub-σ-fields F_n. If X is a Banach space, we shall say that $\boldsymbol{x} = (x_n)$ is an X-valued martingale if each x_n is an F_n-measurable X-valued Bochner integrable function and if $\int_A x_n dP = \int_A x_{n+1} dP$ for each A in F_n and each n; we shall *not* suppose that $x_0 = 0$. We set $d_0 = x_0$, and $d_n = x_n - x_{n-1}$ for $n \geqq 1$.

If the sequence (F_n) is generated by a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of symmetric Bernoulli random variables, we shall say that $\boldsymbol{x}$ is a Walsh–Paley martingale. In this case, we shall write Δ for Ω, and consider $\Delta = \lim \Delta_n$ as a projective limit of finite fields in the usual way.

3. Some spaces of martingales

In this section, we shall suppose that X is a Banach space whose dual X^* has the Radon–Nikodym property, and that δ is an Orlicz M-function, satisfying the Δ_2-condition, with conjugate Orlicz function ρ.

Let $M(X)$ denote the space of all X-valued martingales (on $(\Omega, F, (F_n), P)$). If $\boldsymbol{x} \in M(X)$, let $D(\boldsymbol{x}) = (d_n)$. D can be considered as a linear map from M into the measurable functions on Φ, where Φ is the disjoint union of the spaces (Ω, F_n, P) $(N = 0, 1, 2, \cdots)$ (or, when $\Omega = \Delta$, the disjoint union of the fields Δ_n).

Let

$$M_\delta(X) = \{\boldsymbol{x} : D(\boldsymbol{x}) \in L_\delta(\Phi, X)\}$$
$$= \{\boldsymbol{x} : \Sigma E\delta(\| d_n \|) < \infty.$$

We give $M_\delta(X)$ the norm induced by the mapping D and the norm on $L_\delta(\Phi, X)$.

Now suppose that φ is a continuous linear functional on $M_\delta(X)$. By the Hahn–Banach theorem, there exists an element $\psi \in (L_\delta(\Phi, X))^* = L_\rho(\Phi, X^*)$ such that

$$\varphi(\boldsymbol{x}) = \psi(D(\boldsymbol{x})) \quad \text{for } \boldsymbol{x} \text{ in } M_\delta(X)$$

and $\| \varphi \| = \| \psi \|$. ψ is not unique, nor does it necessarily belong to $D(M_\rho(X^*))$. At the cost of losing norm equality, we can rectify this. We need an elementary lemma.

LEMMA 1. *Suppose that (Ω, F, P) is a probability space and F_0 a sub-σ-field of F. Let $H = \{f \in L_\delta(X) \colon E(f \mid F_0) = 0\}$. Then if g_1 and g_2 are in $L_\rho(X^*)$,*

$$E(g_1(h)) = E(g_2(h)) \quad \textit{for all } h \textit{ in } H$$

if and only if

$$g_1 - E(g_1 \mid F_0) = g_2 - E(g_2 \mid F_0).$$

PROOF. Suppose that $E(g_1(h)) = E(g_2(h))$ for all h in H. If $f \in L_\delta(X)$, $f - E(f \mid F_0) \in H$, so that

$$E(g_1(f - E(f \mid F_0))) = E(g_2(f - E(f \mid F_0))).$$

But

$$E(g_i(E(f \mid F_0))) = E(E(g_i \mid F_0)(f))$$

for $i = 1, 2$, so that

$$E((g_1 - E(g_1 \mid F_0))(f)) = E((g_2 - E(g_2 \mid F_0))(f)).$$

Conversely if $h \in H$,

$$E(g_i(h)) = E((g_i - E(g_i \mid F_0))(h))$$

for $i = 1, 2$, so that if $g_1 - E(g_1 \mid F_0) = g_2 - E(g_2 \mid F_0)$, $E(g_1(h)) = E(g_2(h))$ for all h in H.

Consequently there exists a unique $x^* \in M_\rho(X^*)$ such that

$$\varphi(x) = D(x^*)(D(x))$$

$$= d_0^*(d_0) + \sum_{i=1}^{\infty} E(d_i^*(d_i)).$$

Since $d_0^* = \psi_0$ and $d_i^* = \psi_i - E(\psi_i \mid F_{i-1})$ for $i \geqq 1$, it follows that $\| \varphi \| \leqq \| x^* \|_\rho \leqq 2\| \varphi \|$. Conversely each element of $M_\rho(X^*)$ determines an element of $M_\delta(X)^*$ in this way. Thus we have

THEOREM 1. *There is a natural isomorphism between $M_\rho(X^*)$ and $M_\delta(X)^*$. If $x^* = (x_n^*)$ corresponds to the functional φ, $\| \varphi \| \leqq \| x^* \|_\rho \leqq 2\| \varphi \|$.*

We shall also be concerned with the space $M^\infty(X)$ of X-valued closed uniformly bounded martingales. If $x = (x_n) \in M^\infty(X)$ let us denote the closure of x_n by x_∞. The map $x \to x_\infty$ is of course a linear isomorphism of $M^\infty(X)$ onto the space $L_b^\infty(X)$ of essentially uniformly bounded Bochner measurable functions. We give $M^\infty(X)$ the norm $\| x \|_\infty = \| x_\infty \|_\infty$, so that $M^\infty(X)$ is a Banach space. Note that if $x \in M^\infty(X)$, $E(\| x_\infty \|) = \sup_n E(\| x_n \|)$.

4. Uniform convexity and martingale inequalities

We now turn to martingale inequalities. The first result is a conditional version of [4, prop. 2.1]. If $\boldsymbol{x} = (x_n)$ is a martingale with increments (d_n) and if $p \geqq 1$, let us set

$$m_p(d_0) = \| d_0 \|,$$

$$m_p(d_n) = (E(\| d_n \|^p \mid F_{n-1}))^{1/p} \quad \text{for } n \geqq 1.$$

Note that if $\boldsymbol{x}$ is a Walsh–Paley martingale $m_p(d_n) = \| d_n \|$.

THEOREM 2. *If $C > 0$ and $1 < p \leqq 2$ there exists a constant $k_{C,p} > 0$ such that*

$$k_{C,p} \sum_{i=0}^{\infty} E(\delta_X(k_{C,p} m_p(d_i))) \leqq \sup_n \| x_n \|_p^p$$

for any X-valued martingale (x_n) with $\sup_n \| x_n \|_\infty \leqq C$.

PROOF. By the remarks in Section 2, there exist $l > 0$ and a function j such that

(i) $g(\varepsilon) = j(\varepsilon^{1/p})$ is convex,

(ii) $j(\varepsilon) \geqq l\delta_X(l\varepsilon)$, for all $0 < \varepsilon \leqq 2$, and

(iii) $\delta_{L^p(X)}(\varepsilon) \geqq lj(l\varepsilon)$, for all $0 < \varepsilon \leqq 2$.

Note that $(\| x_n \|^p)$ is a submartingale, so that if $A \in F_n$ and $\int_A \| x_{n+1} \|^p dP = \lambda^p > 0$, $\int_A \| x_n \|^p dP \leqq \int_A \| x_n + \frac{1}{2} d_{n+1} \|^p dP \leqq \lambda^p$. Thus

$$\begin{aligned} \lambda^{-p} \int_A \| x_n \|^p dP &\leqq \lambda^{-1} \| x_n \chi_A \|_p \\ &\leqq \lambda^{-1} \| (x_n + \tfrac{1}{2} d_{n+1}) \chi_A \|_p \\ &= \| \tfrac{1}{2}(\lambda^{-1} x_n \chi_A + \lambda^{-1} x_{n+1} \chi_A \|_p \\ &\leqq 1 - lj(l\lambda^{-1} \| d_{n+1} \chi_A \|_p). \end{aligned}$$

In other words,

$$\begin{aligned} \int_A \| x_{n+1} \|^p dP - \int_A \| x_n \|^p dP &\geqq \lambda^p lg[l^p \lambda^{-r} \| d_{n+1} \chi_X \|_p^p] \\ &= lC^p P(A) \left(\frac{\lambda^p}{C^p P(A)} \right) g \left[l^p \left(\frac{C^p P(A)}{\lambda^p} \right) \frac{\int_A m_p^p(d_{n+1}) dP}{C^p P(A)} \right] \\ &\geqq lC^p P(A) g \left[\frac{l^p}{C^p} \frac{\int_A m_p^p(d_{n+1}) dP}{P(A)} \right]. \end{aligned}$$

Now fix $\varepsilon > 0$ and let

$$A_k = \{\omega : (k-1)\varepsilon \leqq g[l^p C^{-p} m_p^p(d_{n+1})(\omega)] < k\varepsilon\}.$$

Then if $P(A_k) > 0$,

$$g\left[\frac{l^p}{C^p}\frac{\int_{A_k} m_p^p(d_{n+1})dP}{P(A_k)}\right] \geqq (k-1)\varepsilon,$$

so that, adding over k, we obtain that

$$\| x_{n+1} \|_p^p - \| x_n \|_p^p \geqq lC^p \sum_{k=1}^{\infty} (k-1)\varepsilon P(A_k).$$

On the other hand,

$$\int_{A_k} g[l^p C^{-p} m_p^p(d_{n+1})]dP \leqq k\varepsilon P(A_k),$$

so that

$$\| x_{n+1} \|_p^p - \| x_n \|_p^p \geqq lC^p(E(g[l^p C^{-p} m_p^p(d_{n+1})]) - \varepsilon).$$

Since ε is arbitrary we get that

$$\| x_{n+1} \|_p^p - \| x_n \|_p^p \geqq l^2 C^p E(\delta_X(l^2 C^{-1} m_p(d_{n+1}))).$$

Further, since $\delta_X \overset{0}{<} \varepsilon^p$,

$$\| x_0 \|_p^p = \| d_0 \|_p^p \geqq k\delta_X(km_p(d_0)),$$

for some suitable k. Adding, we obtain the required result.

We now turn to the problem of renorming a Banach space with an equivalent δ-uniformly convex norm.

THEOREM 3. *Suppose that $(X, \| \ \|)$ is a Banach space and that δ is an Orlicz M-function which satisfies the Δ_2-condition. Then the following are equivalent.*

(i) *There is an equivalent norm on X under which X is δ-uniformly convex.*

(ii) *For each $1 < p \leqq 2$ and each $C > 0$ there exists a constant $K > 0$ such that*

$$\sum_{i=0}^{\infty} E(\delta(m_p(d_i))) \leqq K \sup \| x_n \|_p^p,$$

for any X-valued martingale (x_n) with $\sup_n \|x_n\|_\infty \leqq C$.

(iii) *For some $1 < p \leqq 2$ and some $C > 0$ there exists a constant $K > 0$ such that*

$$\sum_{i=0}^{\infty} E(\delta(\| d_i \|)) \leqq K \sup_n \| x_n \|_p^p,$$

for any X-valued Walsh–Paley martingale (x_n) with $\sup_n \| x_n \|_\infty \leqq C$.

(iv) *If* (x_n) *is any closed* X*-valued Walsh–Paley martingale with* $\sup \| x_n \|_\infty < \infty$, *then* $\sum_{i=0}^{\infty} E(\delta(\| d_i \|)) < \infty$.

PROOF. Since δ satisfies the Δ_2-condition, (i) implies (ii), by Theorem 2. Since $m_p(d_i) = \| d_i \|$ for a Walsh–Paley martingale, (ii) implies (iii), and, using the Δ_2-condition again, it is easy to see that (iii) implies (iv).

Let us now suppose that (iv) holds, and construct an equivalent norm on X under which X is δ-uniformly convex. The construction is modelled on that used by Pisier [4].

Condition (iv) says that, restricting attention to the probability space Δ, $M^\infty(X) \subseteqq M_\delta(X)$. Since the inclusion has a closed graph, it is continuous. Thus there exists $\theta > 0$ such that if $\| \mathbf{x} \|_\infty \leqq \theta$, $\| D(\mathbf{x}) \|_\delta \leqq 1$; i.e. $\sum_{i=0}^{\infty} E(\delta(\| d_i \|)) \leqq 1$. Using the Δ_2-condition it follows that there exists $K > 0$ such that if $\| \mathbf{x} \|_\infty \leqq 1$, then $\sum_{i=0}^{\infty} E(\delta(\| d_i \|)) \leqq K$. Now if $0 < \alpha = \| \mathbf{x} \|_\infty \leqq 1$,

$$\sum_{i=0}^{\infty} E\left(\frac{1}{\alpha}\delta(\| d_i \|)\right) \leqq \sum_{i=0}^{\infty} E\left(\delta\left(\frac{\| d_i \|}{\alpha}\right)\right) \leqq K,$$

so that $\sum_{i=0}^{\infty} E(\delta(\| d_i \|)) \leqq K \| \mathbf{x} \|_\infty$.

Now if $x \in X$ and $\| x \| = 1$, define

$$\gamma(x) = \inf \left[2(K+1)E(\| x_\infty \|) - (K+1)^{-1} \sum_{n=1}^{\infty} E\delta(\| d_n \|) \right]$$

where the infimum is taken over all Walsh–Paley martingales $\mathbf{x}$ in $M^\infty(X)$ with $\| \mathbf{x} \|_\infty \leqq 1$ and $x_0 = x$.

If we take $x_n = x$ for all n it follows that

$$\text{(a)} \qquad \gamma(x) \leqq 2(K+1)\| x \|.$$

If $\| x \| \geqq 1/2(K+1)$, $E(\| x_\infty \|) \geqq 1/2(K+1)$ while $\sum_{n=1}^{\infty} E(\delta(\| d_n \|)) \leqq K$, so that

$$\gamma(x) \geqq 2K/2(K+1) + 2E(\| x_\infty \|) - K/(K+1)$$

so that

$$\text{(b)} \qquad \gamma(x) \geqq 2\| x \| \quad \text{for} \quad \| x \| \geqq 1/2(K+1).$$

Now suppose that $\gamma(x) \leqq 1$ and $\gamma(y) \leqq 1$. Let $z = \frac{1}{2}(x+y)$ and let $\eta > 0$. Then there exist Walsh–Paley martingales $\mathbf{x}$ and $\mathbf{y}$ in $M^\infty(X)$ with $\| \mathbf{x} \|_\infty \leqq 1$, $\| \mathbf{y} \|_\infty \leqq 1$, $x_0 = x$, $y_0 = y$ such that, setting $\rho_n = y_n - y_{n-1}$,

$$\gamma(x) + \eta \geqq 2(K+1)E(\| x_\infty \|) - (K+1)^{-1} \sum_{n=1}^{\infty} E(\delta(\| d_n \|)),$$

$$\gamma(y)+\eta \geqq 2(K+1)E(\|y_\infty\|)-(K+1)^{-1}\sum_{n=1}^{\infty} E(\delta(\|\rho_n\|)).$$

We now consider the martingale obtained by setting $z_0 = z$,

$$z_1 = z + \varepsilon_1\left(\frac{x-y}{2}\right) = \left(\frac{1+\varepsilon_1}{2}\right)x + \left(\frac{1-\varepsilon_1}{2}\right)y$$

and

$$z_n(\varepsilon_1, \varepsilon_2, \cdots) = \left(\frac{1+\varepsilon_1}{2}\right)x_{n-1}(\varepsilon_2, \varepsilon_3, \cdots) + \left(\frac{1-\varepsilon_1}{2}\right)y_{n-1}(\varepsilon_2, \varepsilon_3, \cdots)$$

for $n \geqq 2$. Note that

$$z_n(1, \varepsilon_2, \varepsilon_3, \cdots) = x_{n-1}(\varepsilon_2, \varepsilon_3, \cdots),$$

and

$$z_n(-1, \varepsilon_2, \varepsilon_3, \cdots) = y_{n-1}(\varepsilon_2, \varepsilon_3, \cdots)$$

so that

$$z_\infty(1, \varepsilon_2, \varepsilon_3, \cdots) = x_\infty(\varepsilon_2, \varepsilon_3, \cdots)$$

and

$$z_\infty(-1, \varepsilon_2, \varepsilon_3, \cdots) = y_\infty(\varepsilon_2, \varepsilon_3, \cdots).$$

Consequently $E(\|z_\infty\|) = \frac{1}{2}(E(\|x_\infty\|) + E(\|y_\infty\|))$ and, setting $f_n = z_n - z_{n-1}$,

$$\begin{aligned}\sum_{n=1}^{\infty} E(\delta(\|f_n\|)) &= \delta(\|x-y\|/2) + \sum_{n=2}^{\infty} E(\delta(\|f_n\|)) \\ &= \delta(\|x-y\|/2) + \tfrac{1}{2}\sum_{n=1}^{\infty} E(\delta(\|d_n\|)) + \tfrac{1}{2}\sum_{n=1}^{\infty} E(\delta(\|\rho_n\|)).\end{aligned}$$

Thus

$$\gamma(z) \leqq \tfrac{1}{2}(\gamma(x)+\gamma(y)) + \eta - (K+1)^{-1}\delta(\|x-y\|/2).$$

Since η is arbitrary we have

(c) $$\gamma\left(\frac{x+y}{2}\right) \leqq \tfrac{1}{2}(\gamma(x)+\gamma(y)) - (K+1)^{-1}\delta(\|x-y\|/2).$$

In particular, $B = \{x : \gamma(x) \leqq 1\}$ is an absolutely convex subset of $\{x : \|x\| \leqq 1\}$. It follows from (a) that $B \supseteq \{x : \|x\| \leqq 1/2(K+1)\}$ and from (b) that $B \subseteq \{x : \|x\| \leqq 1/2\}$. Thus the gauge of B, $|\ |_B$ say, is a norm on X equivalent to the original norm.

Now suppose that $|x|_B = |y|_B = 1$; i.e. $\gamma(x) = \gamma(y) = 1$. By (a), $\|x\| \geqq 1/2(K+1)$ and $\|y\| \geqq 1/2(K+1)$, and so by (b), $\|x\| \leqq 1/2$ and $\|y\| \leqq 1/2$; thus $\|z\| \leqq 1/2$.

Since δ satisfies the Δ_2-condition, there exists $\theta > 0$ such that $\delta((1-\lambda)t) \geqq (1-\theta\lambda)\delta(t)$ for all t, and $0 < \lambda \leqq 1/2$. Let $\varepsilon^{-1} = (2(K+1)^2 + \theta K)(1+K)$, and let $\alpha = \varepsilon\delta(\|x-y\|/2)$. Since $\delta(1/2) \leqq K/2$ (this follows easily from the basic property of K), $\alpha \leqq 1/2$.

Now let $w_0 = w = (1+\alpha)z$ and let $w_n = w_0 + (1-\alpha)(z_n - z_0) = 2\alpha z + (1-\alpha)z_n$ for $n \geqq 1$. $\|w\| = (1+\alpha)\|z\| \leqq 1$, so that $\gamma(w)$ is defined. Further

$$\|w_\infty\|_\infty \leqq 2\alpha\|z\| + (1-\alpha)\|z_\infty\|_\infty \leqq 1.$$

Also $E(\|w_\infty\|) \leqq 2\alpha\|z\| + (1-\alpha)E(\|z_\infty\|) \leqq (1+\alpha)E(\|z_\infty\|)$ and, setting $g_n = w_n - w_{n-1} = (1-\alpha)f_n$,

$$E(\delta(\|g_n\|)) \geqq (1-\theta\alpha)E(\delta(\|f_n\|)).$$

Combining these, we see that

$$\begin{aligned}\gamma(w) &\leqq 1 + \eta - (K+1)^{-1}\delta(\|x-y\|/2) \\ &\quad + 2\alpha(K+1)E(\|z_\infty\|) + \frac{\theta\alpha}{K+1}\sum_{n=1}^{\infty} E(\delta(\|f_n\|)), \\ &\leqq 1 + \eta - (K+1)^{-1}\delta(\|x-y\|/2) + 2\alpha(K+1) + \frac{\theta\alpha K}{K+1}.\end{aligned}$$

Since η is arbitrary, it follows from the definition of α that $\gamma(w) \leqq 1$. Thus

$$|z|_B \leqq (1+\alpha)^{-1} \leqq 1 - \frac{\varepsilon}{2}\delta(\|x-y\|/2).$$

Since $\|\ \|$ is equivalent to $|\ |_B$, this shows that $(X, |\ |_B)$ is δ-uniformly convex.

5. Uniform smoothness and martingale inequalities

We now use duality to obtain the result for smoothness corresponding to Theorem 3.

THEOREM 4. *Suppose that $(X, \|\ \|)$ is a Banach space and that ρ is an Orlicz M-function which satisfies the Δ_2-condition. Then the following are equivalent*:

(i) *There is an equivalent norm on X under which X is ρ-uniformly smooth.*

(ii) *If (x_n) is any X-valued Walsh–Paley martingale with $\sum_{i=0}^{\infty} E\rho(\|d_i\|) < \infty$, then $\sup_n E(\|x_n\|) < \infty$.*

(iii) *If (x_n) is any X-valued Walsh–Paley martingale with $\sum_{i=0}^{\infty} E\rho(\| d_i \|) < \infty$, then x_n converges in $L^1(X)$, and almost surely.*

PROOF. Suppose that (i) holds, and that (x_n) is an X-valued Walsh–Paley martingale with $\sum_{i=0}^{\infty} E\rho(\| d_i \|) < \infty$. Since X^* is δ-uniformly convex (where δ is the Orlicz function conjugate to ρ), as in Theorem 3 there exists $K \geqq 1$ such that

$$\sum E\delta(\| d_n^* \|) \leqq K \sup_n \| x_n^* \|_\infty$$

for all uniformly bounded X^*-valued Walsh–Paley martingales (x_n^*) with $\sup_n \| x_n^* \|_\infty \leqq 1$. As

$$\sup_n E(\| x_n \|) = \sup\{\sup_n E(x_n^*(x_n)) : (x_n^*) \in M_\infty(X^*) : \|(x_n^*)\| \leqq 1\}$$

we have that

$$\begin{aligned} \sup_n E(\| x_n \|) &\leqq K \sup\left\{ \sup_n E(x_n^*(x_n)) : \sum E\delta(\| d_n^* \|) \leqq 1 \right\} \\ &= K \sup\left\{ \sup_n \sum_{j=0}^{\infty} Ed_j^*(d_j) : \sum E\delta(\| d_n^* \|) \leqq 1 \right\} \\ &= K \| D(x^*) \|_\delta < \infty. \end{aligned}$$

Thus (i) implies (ii). A standard argument shows that (ii) implies (iii).

Finally suppose that (iii) holds. If $\boldsymbol{x} \in M^\rho(X)$ (with respect to $\Delta, (\Delta_n)$), let $T(\boldsymbol{x}) = \lim x_n$. T maps $M_\rho(X)$ into $L^1(X)$, and, again by the closed graph theorem, T is continuous. The transposed mapping maps $(L^1(X))^*$ into $M_\delta(X^*)$, and therefore maps $L_\delta^\infty(X^*)$ into $M_\delta(X^*)$. If $f \in L_\delta^\infty(X^*)$, $(T^*(f))_n = E(f \mid \Delta_n)$, so that $M^\infty(X^*) \subseteq M_\delta(X^*)$. Consequently X^* can be renormed to be δ-uniformly convex, by Theorem 3, and so (i) holds.

REFERENCES

1. T. Figiel, *On the moduli of convexity and smoothness*, Studia Math. **56** (1976), 121-155.

2. K. Lindberg, *On subspaces of Orlicz sequence spaces*, Studia Math. **45** (1973), 119-146.

3. G. Pisier, *Martingales à valeurs dans les espaces uniformément convexes*, C. R. Acad. Sci. Paris Ser. A **279** (1974), 647-649.

4. G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), 326-350.

ST. JOHN'S COLLEGE
CAMBRIDGE, ENGLAND